WSU Economics PhD Mathcamp Notes

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1 Real Analysis

1.1 Monotonic Sequences

A sequence is increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and is decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotonic** if it is either increasing or decreasing.

2 Topology

2.1 Open Sets

An open ball of radius $\varepsilon > 0$ centered about point $x \in X$ can be defined by:

$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$$

If we a working in the metric space $(\mathbb{R}, d_1)$, then the open ball is just an open interval, and is usually referred to as an ε-neighborhood. In the metric space $(\mathbb{R}^2, d_2)$, the open ball is a circle, and in the metric space $(\mathbb{R}^3, d_2)$, the open ball is a sphere.

A set $A \subseteq X$ is open if for all points $a \in A$ if there exists an open ball $B_\varepsilon(a) \subseteq A$.

Practice

Using the definition above and assuming the metric space is $(\mathbb{R}, d_1)$:

1. Show that $B_\varepsilon(a)$ is an open set.

   From the definition of open set, we need to show that for every point in $B_\varepsilon(a)$, we can make an open ball around any point, and that open ball must be contained in $B_\varepsilon(a)$. Consider $y \in B_\varepsilon(a)$. If we define an open ball around y as $B_{\varepsilon_1}(y)$, where $\varepsilon_1 = \varepsilon - |y - a|$, you'll see that $B_{\varepsilon_1}(y) \subseteq B_\varepsilon(a)$. Thus $\subseteq B_\varepsilon(a)$. Hence $B_\varepsilon(a)$ is open.

2. Show that $\mathbb{R}$ is an open set.

   Pick a $y \in$, and put a ball of radius $\varepsilon \in$ around y. Notice that since $\varepsilon \in$, it is always that case that $B_\varepsilon(y) \in$. Thus is open.

3. Show that $(0, 1)$ is an open set.

   See 2018 Assignment 3 solutions.

The following theorems hold for open sets:

1. The union of open sets is open.
2. The finite intersection of open sets is open.

2.2 Closed Sets

Let (X, d) be a metric space, and S be a subset of X. A point x is a **limit point** of set S iff $\{B_\varepsilon(x) - \{x\}\} \cap S \neq \emptyset$ where $\varepsilon > 0$. For x to be a limit point of S, if we take an open ball around it, no matter what the size of that open ball, there will exist other points from S other than x in that open ball. Note: for a point x to be a limit point of S, it doesn't necessarily have to be an element of S. It only has to contain an element from S in its open ball for every $\varepsilon > 0$. The set of all limit points of a set S is usually denoted as S'

We say that a set S is **closed** iff S contains all of its limit points. In other words, $S' \subseteq S$. It does not have to be the case that $S = S'$.

Practice

The set of natural numbers, $\mathbb{N}$, can be written in the form: $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup ...$ where $\{n\}$ is said to be an isolated point. Is $\{n\}$ a limit point? What does that tell us about the set $\mathbb{N}$, is it open, closed, or neither.

$\{n\}$ where $n \in$ is not a limit point as we can easily find an $\varepsilon > 0$ such that a ball around every point in the set contains only that point. Thus there are no limit points in the set . Notice that, trivially, contains all of its limit points, so is closed.

The following theorems hold for closed sets:

1. The finite union of closed sets is closed.
2. The intersection of closed sets is closed.

2.3 Open and Closed Sets

The following theorem is useful for determining if a set is open or closed: The compliment of a closed set is open, and the compliment of an open set is closed.

Practice

1. Show that the empty set, $\emptyset$, is both closed and open.

   For the empty set, the set of its limit points is just the empty set. Since $\emptyset \subseteq \emptyset$, then the $\emptyset$ is closed.

   Notice that $= \emptyset$. We see that contains all of its limit points, thus it is closed. Then $\emptyset$ is open.

2. Determine if $[0, 1] \cup \{2\}$ is open, closed, or neither.

Notice that the set of limit points for $[0,1] \cup \{2\}$ is $[0,1]$. Since $[0,1] \subseteq [0,1] \cup \{2\}$, then $[0,1] \cup \{2\}$ is closed.

2.4 Compact Sets

A subset S in a Euclidean space is said to be compact iff S is closed and bounded.

A subset S is compact if every sequence in S has a subsequence that converges to a point in S.

Practice

Show that for a compact set $S \subseteq \mathbb{R}$, the supremum and infimum or S are elements of S.

Let $S \subseteq R$. Suppose that S be bounded and let $b = \sup S$. For every $\varepsilon > 0$, there exists an $s \in S$ such that $b - \varepsilon < s$. Notice that we have defined an open ball $B_\varepsilon(b)$, and we see that $\exists s \in B_\varepsilon(b)$ for any $\varepsilon > 0$. Thus b is a limit point of S. Since S is closed, S must contain all of its limit points. Therefore $b \in S$. Or in other words, $\sup S \in S$.

A similar argument can be used to show that $\inf S \in S$.

3 Advanced Theorems

3.1 Continuous Functions

A function $f : A \to \mathbb{R}$ is **continuous** at $c \in A$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that when $|x - c| < \delta$, it follows that $|f(x) - f(c)| < \varepsilon$. f is said to be continuous on A if f is continuous at every point in the domain A.

Additional properties: Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be continuous at a point $c \in A$. Then:

1. $kf(x)$ is continuous at c for every $k \in \mathbb{R}$
2. $f(x) + g(x)$ is continuous at c.
3. $f(x) \cdot g(x)$ is continuous at c.
4. $\frac{f(x)}{g(x)}$ is continuous at c, given $\frac{f(x)}{g(x)}$ exists.

3.2 Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and r is a real number such that $f(a) \leq r \leq f(b)$ or $f(z) \geq r \geq f(b)$, then there exists a $c \in [a, b]$ such that $f(c) = r$.

Exercises

Consider the metric space $(\mathbb{R}, d_1)$ for the following problems, where d_1 is the absolute value metric.

1. Using the definition of open ball (or ε-neighborhood). Show that (a, b), where $a < b$, is an open set.

2. Let

$$B = \left\{ \frac{(-1)^n n}{n+1} : n \in \mathbb{N} \right\}$$

   (a) Find the limit points of B.

   (b) Is B a closed set? Why or why not?

   (c) Is B an open set? Why or why not?