Optimization and Multivariate Calculus

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These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Optimization

1.1 Constrained Optimization

We have covered unconstrained optimization, and will now consider optimizing objective functions that are subject to constraints. The general form of a constrained optimization problem is:

$$\max_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } g_1(\mathbf{x}) \leq b_1, \ldots, g_k(\mathbf{x}) \leq b_k,$$
$$h_1(\mathbf{x}) = c_1, \ldots, h_m(\mathbf{x}) = c_m$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

The function f is known as the objective function, $g_1, \ldots, g_k$ are inequality constraints, and $h_1, \ldots, h_m$ are equality constraints. In some cases, additional constraints $x_1 \geq 0, \ldots, x_n \geq 0$ known as nonnegativity constraints could also be included.

1.2 Equality Constraints

Let's start with the following constrained optimization problem:

$$\max_{x_1, x_2} U(x_1, x_2) \text{ subject to } p_1x_1 + p_2x_2 = I$$

This example might look a bit familiar. When you start looking at utility maximization problems, this is usually the one you start with. x_1 and x_2 are two goods that enter into this particular utility function $U(x_1, x_2)$, which is our objective function. Notice that the (budget) constraint $p_1x_1 + p_2x_2 = I$ is of the form $h(x_1, x_2) = I$. The optimal bundle of goods $\mathbf{x}^* = (x_1^*, x_2^*)$ will be the bundle on the constraint where the utility function is maximized. In order to find the optimal bundle of goods, $\mathbf{x}^* = (x_1^*, x_2^*)$, we find the point where the budget constraint is tangent to the utility function. Or in other words:

$$-\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}}(\mathbf{x}^*) = -\frac{\frac{\partial h}{\partial x_1}}{\frac{\partial h}{\partial x_2}}(\mathbf{x}^*)$$

Rearranging the function above yields the following equality:

$$\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial h}{\partial x_1}}(\mathbf{x}^*) = \frac{\frac{\partial U}{\partial x_2}}{\frac{\partial h}{\partial x_2}}(\mathbf{x}^*)$$

Now let μ equal these values:

$$\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial h}{\partial x_1}}(\mathbf{x}^*) = \frac{\frac{\partial U}{\partial x_2}}{\frac{\partial h}{\partial x_2}}(\mathbf{x}^*) = \mu$$

μ (known as a Lagrange multiplier) is thus the rate of change in optimal output resulting from the change of the constraint (I). We can rewrite the equalities from above to give us:

$$\frac{\partial U}{\partial x_1}(\mathbf{x}^*) - \mu\frac{\partial h}{\partial x_1}(\mathbf{x}^*) = 0$$
$$\frac{\partial U}{\partial x_2}(\mathbf{x}^*) - \mu\frac{\partial h}{\partial x_2}(\mathbf{x}^*) = 0$$

This gives us two equations with three unknowns (x_1, x_2, and μ). Adding the constraint gives us another equation:

$$\frac{\partial U}{\partial x_1}(\mathbf{x}^*) - \mu\frac{\partial h}{\partial x_1}(\mathbf{x}^*) = 0 \quad (1)$$
$$\frac{\partial U}{\partial x_2}(\mathbf{x}^*) - \mu\frac{\partial h}{\partial x_2}(\mathbf{x}^*) = 0 \quad (2)$$
$$h(x_1^*, x_2^*) = I \quad (3)$$

We can represent the constrained optimization problem as one function (called the Lagrangian function):

$$L(x_1, x_2, \mu) = U(x_1, x_2) - \mu(h(x_1, x_2) - I)$$

We have reduced this constrained optimization down to one function, but at the cost of introducing one more variable (μ). Notice that taking the first order conditions of the Lagrangian yield same equations, namely equations (1) - (3), that we derived above. Solving for (x^*, y^*, z^*) using these FOCs give us critical points of the Lagrangian.
We can then extend this idea by adding more equality constraints. Consider the optimization problem that has m equality constraints:

$$\max_{\mathbf{x}} U(\mathbf{x}) \text{ subject to } h_1(\mathbf{x}) = c_1, \ldots, h_m(\mathbf{x}) = c_m$$

The Lagrangian becomes:

$$L(x_1, x_2, \mu_1, \ldots, \mu_n) = U(x_1, x_2) - \mu_1(h_1(x_1, x_2) - c_1) - \ldots - \mu_m(h_m(x_1, x_2) - c_m)$$

The first order conditions are thus:

$$\frac{\partial L}{\partial x_1} : \frac{\partial U}{\partial x_1}(\mathbf{x}^*) - \mu\frac{\partial h}{\partial x_1}(\mathbf{x}^*) = 0$$
$$\frac{\partial L}{\partial x_2} : \frac{\partial U}{\partial x_2}(\mathbf{x}^*) - \mu\frac{\partial h}{\partial x_2}(\mathbf{x}^*) = 0$$
$$\frac{\partial L}{\partial \mu_i} : h_i(\mathbf{x}^*) = c_i \text{ for } i = 1, 2, \ldots, n$$

Solve the following utility max problem:

$$\max_{x_1, x_2} x_1x_2 \text{ subject to } x_1 + 4x_2 = 16$$

First let's form the Lagrangian:

$$L(x_1, x_2, \mu) = x_1x_2 - \mu(x_1 + 4x_2 - 16)$$

The first order conditions are:

$$\frac{\partial L}{\partial x_1} : x_2 - \mu = 0$$
$$\frac{\partial L}{\partial x_2} : x_1 - 4\mu = 0$$
$$\frac{\partial L}{\partial \mu} : -(x_1 + 4x_2 - 16) = 0$$

Solving for μ in the first FOC gives us $\mu = x_2$. Plugging this into the second FOC yields: $x_1 = 4x_2$. Now we can plug this into the third FOC (the budget constraint) to get $x_1 = 8$, $x_2 = 2$, $\mu = 2$.

$(8, 2)$ is the arg max for this utility max problem.

1.3 Inequality Constraints

Now we focus on the case when the constraints are inequalities. Consider the following optimization problem with one inequality constraint:

$$\max_{\mathbf{x}} U(\mathbf{x}) \text{ subject to } g(\mathbf{x}) \leq b$$

The Lagrangian function is thus:

$$L(\mathbf{x}, \lambda) = U(\mathbf{x}) - \lambda[g(\mathbf{x}) - b]$$

With the inequality constraint, $g(\mathbf{x}) \leq b$, it could either be the case that $g(\mathbf{x}) = b$ (the constraint is binding), or $g(\mathbf{x}) < b$ (the constraint is not binding). Notice that when the constraint is not binding, the constrained critical point will be the same as the unconstrained critical point. As opposed to the optimization with equality constraints, the first order condition for the Lagrange multiplier will not be part of our calculations for $\mathbf{x}^*$. However, we can use the complementary slackness condition to solve for $\mathbf{x}^*$:

$$\lambda[g(\mathbf{x}) - b] = 0$$

Notice that either $\lambda = 0$, or $g(\mathbf{x}) - b = 0$. So the equations we can use to solve for $\mathbf{x}^*$ are:

$$\frac{\partial L}{\partial x_1} : \frac{\partial U}{\partial x_1}(\mathbf{x}^*) - \mu \frac{\partial h}{\partial x_1}(\mathbf{x}^*) = 0$$
$$\frac{\partial L}{\partial x_2} : \frac{\partial U}{\partial x_2}(\mathbf{x}^*) - \mu \frac{\partial h}{\partial x_2}(\mathbf{x}^*) = 0$$
$$\lambda^*[g(\mathbf{x}^*) - b] = 0$$
$$\lambda^* \geq 0$$
$$g(\mathbf{x}^*) \leq b$$

1.4 Karush Kuhn Tucker Conditions

Extending the previous optimization problem, we now consider the following optimization problem. Optimization problems of this form are very common economics:

$$\max_{\mathbf{x}} U(\mathbf{x}) \text{ subject to } g_1(\mathbf{x}) \leq b_1, \ldots, g_k(\mathbf{x}) \leq b_k,$$
$$x_1 \geq 0, \ldots, x_n \geq 0$$

The Lagrangian function is thus:

$$L(\mathbf{x}, \lambda_1, \ldots, \lambda_k, \nu_1, \ldots, \nu_n) = U(\mathbf{x}) - \lambda_1[g_1(\mathbf{x}) - b_1] - \ldots - \lambda_1[g_1(\mathbf{x}) - b_1] + \nu_1 x_1 + \ldots + \nu_n x_n$$

The Karush Kuhn Tucker conditions are then:

$$\begin{aligned}
&\frac{\partial L}{\partial x_1}: \frac{\partial U}{\partial x_1}(\mathbf{x}^*) - \mu \frac{\partial h}{\partial x_1}(\mathbf{x}^*) = 0\\
&\frac{\partial L}{\partial x_2}: \frac{\partial U}{\partial x_2}(\mathbf{x}^*) - \mu \frac{\partial h}{\partial x_2}(\mathbf{x}^*) = 0\\
&\lambda_1^*[g_1(\mathbf{x}^*) - b_1] = 0\\
&\vdots\\
&\lambda_k^*[g_k(\mathbf{x}^*) - b_k] = 0\\
&\nu_1 x_1 = 0\\
&\vdots\\
&\nu_n x_n = 0\\
&\lambda_1^*, \ldots, \lambda_k^*, \nu_1^*, \ldots, \nu_n^* \geq 0\\
&g_1(\mathbf{x}^*) \leq b_1\\
&\vdots\\
&g_k(\mathbf{x}^*) \leq b_k
\end{aligned}$$

Example

Find the Karush Kuhn Tucker (KKT) conditions for the following optimization problem:

$$\begin{aligned}
\max_{x,y,z} xyz \quad \text{subject to } & x + y + z \leq 1\\
& x \geq 0\\
& y \geq 0\\
& z \geq 0
\end{aligned}$$

The Lagrangian is then:

$$L(\mathbf{x}, \lambda, \nu_x, \nu_y, \nu_z) = xyz - \lambda[x + y + z - 1] + \nu_x x + \nu_y y + \nu_z z$$

The KKT conditions are then:

$$\begin{aligned}
&\frac{\partial L}{\partial x}: yz - \lambda + \nu_x = 0\\
&\frac{\partial L}{\partial y}: xz - \lambda + \nu_y = 0\\
&\frac{\partial L}{\partial z}: yz - \lambda + \nu_z = 0\\
&\lambda[x + y + z - 1] = 0\\
&\nu_x x = 0\\
&\nu_y y = 0\\
&\nu_z z = 0\\
&\lambda, \nu_x, \nu_y, \nu_z \geq 0\\
&x + y + z \leq 1
\end{aligned}$$

We could also include the following conditions: $x \geq 0, y \geq 0, z \geq 0$

Exercises

1. The Cobb-Douglas utility function is given by $U(x_1, x_2) = kx_1^{\alpha}x_2^{1-\alpha}$ where x_1 and x_2 are two goods. Assume a consumer's budget set is $p_1x_1 + p_2x_2 \leq I$. Do the following:

   (a) List the Karush Kuhn Tucker conditions for the problem above.

   (b) Solve for x_1, x_2, and the Lagrangian multiplier.

2. Consider the following production function: $y = f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}$ for $i = 1, 2, \ldots, n$ where $a_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. The firm maximizes profits under perfect competition (in other words price or output, $p > 0$, and prices of inputs, $w_i > 0$, are exogenous or given):

$$\max_{x_1, x_2, \ldots, x_n} pf(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} w_i x_i$$

   (a) Solve for the maximizer $(x_1^*, x_2^*, \ldots, x_n^*)$

   (b) Show that x_i^* is homogenous of degree 0 (for prices p and w_i's). What does this mean?