WSU Economics PhD Mathcamp Notes

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These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Matrix Algebra

To add or subtract matrices together, the matrices must be of the same size. The results from these two operations will result in a matrix that is the same size of the matrices operated on. The addition or subtraction of matrices is done entrywise.

Addition

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{bmatrix} \tag{1}$$

Subtraction

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} - \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}-b_{11} & \dots & a_{1n}-b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}-b_{m1} & \dots & a_{mn}-b_{mn} \end{bmatrix} \tag{2}$$

Scalar Multiplication

$$c \cdot \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & ... & ca_{mn} \end{bmatrix} \tag{3}$$

Matrix Multiplication

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{km} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} = \tag{4}$$

$$\begin{bmatrix} a_{11} \cdot b_{11} + ... + a_{1m} \cdot b_{m1} & \dots & a_{11} \cdot b_{1n} + ... + a_{1m} \cdot b_{mn} \\ \vdots & \ddots & \vdots \\ a_{k1} \cdot b_{11} + ... + a_{km} \cdot b_{m1} & \dots & a_{k1} \cdot b_{1n} + ... + a_{km} \cdot b_{mn} \end{bmatrix} \tag{5}$$

Matrix Multiplication Properties

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $c(AB) = (cA)B = A(cB)$
5. $A^k = A \cdot A \cdot \ldots \cdot A$

Transposes

When a matrix is transposed, the rows and columns are interchanged.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}^T = \tag{6}$$

$$\begin{bmatrix} a_{11} & \dots & a_{m1} \\ a_{12} & \dots & a_{m2} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix} \tag{7}$$

Transpose Properties

1. $(AB)^T = B^T A^T$
2. $(A^T)^T = A$
3. $(cA)^T = c(A)^T$
4. $(A + B)^T = A^T + B^T$
5. $(A^T)^{-1} = (A^{-1})^T$
6. $|A^T| = |A|$
7. If A has only real values, then $A^T A$ is positive-semidefinite

Advanced Practice

1. Show that $(AB)^T = B^T A^T \Rightarrow (ABC)^T = C^T B^T A^T$
2. Prove that if A has only real values, then $A^T A$ is positive-semidefinite

Trace

The trace of an $n \times n$ matrix, donated tr, is the sum of the (main) diagonal. If $A = \begin{bmatrix} 3 & 7 \\ 2 & 8 \end{bmatrix}$, then $tr(A) = 11$.

Determinants

It is a bit difficult to describe what a determinant is, but this discussion on stack exchange seems to give the most intuitive idea. A determinant can only be computed for a square matrix. The determinant for a matrix, A, can either be denoted as $|A|$ or $det(A)$.

The determinant of a scalar a is just a.

The determinant of a 2×2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is:

$$a_{11}a_{22} - a_{21}a_{12}$$

The determinant of a 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is:

$$-1^{1+1} \cdot a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + -1^{1+2} \cdot a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + -1^{1+3} \cdot a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Note

We don't have to use the first row to calculate the determinant of a matrix that's bigger than 2×2. For example, if I chose to use the 2nd column, the determinant for the matrix above would now be:

$$-1^{1+2} \cdot a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + -1^{2+2} \cdot a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + -1^{3+2} \cdot a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

If the determinant of a square matrix is nonzero, then that matrix is nonsingular.

Properties

- $|A| = |A^T|$
- $|A||B| = |AB|$

Practice

Use the definition of a determinant for an $n \times n$ matrix to show that the determinant of a 2×2 matrix (which was defined earlier) is equal to $a_{11}a_{22} - a_{21}a_{12}$.

Inverses

An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix B such that:

$$AB = BA = I_n \tag{8}$$

where I_n is an $n \times n$ identity matrix (described in the special matrices section).

Inverse Properties

1. $(A^{-1})^{-1} = A$
2. $(A^T)^{-1} = (A^{-1})^T$
3. $(cA)^{-1} = c^{-1}A^{-1}$
4. If A, B, and C are invertible $n \times n$ matrices, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
5. $|A^{-1}| = |A|^{-1}$
6. $A^{-1}A = AA^{-1} = I$
7. $A^{-1} = \frac{1}{|A|}adj(A)$

The Invertible Matrix Theorem

The following properties for an $n \times n$ matrix A are equivalent (if one is true, all are true; if one is false, all are false):

- A is invertible
- A^T is invertible
- A has n leading coefficients
- There exists a matrix B such that $AB = I$
- There exists a matrix C such that $CA = I$
- The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.
- The equation $Ax = 0$ only has the trivial solution. In other words $x = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}^T$
- A is row equivalent to an $n \times n$ identity matrix
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- A is full rank

One way to find the inverse a matrix is to use the formula below:

$$A^{-1} = \frac{1}{|A|} adj(A)$$

If A is the following matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then:

$$C = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{21} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix} \tag{9}$$

And $adj(A) = C^T$.

Practice 1

Problem 8.22. Note: A^{-2} can also be written as $(A^{-1})^2$
Problem 9.2

Practice 2

Use the method outlined above to invert the following matrix:

$$\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

Practice 3

Use the method outlined above to invert the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{bmatrix}$$

Some Derivatives of Matrices

The following examples show how to take derivatives when matrices are involved (I have only included 2 of the most relevant examples, I encourage you to further explore other properties). Let X be an $n \times k$ matrix, y be a $n \times 1$, and b be $k \times 1$.

1. $\frac{\partial b'X'Xb}{\partial b} = 2X'Xb$

2. $\frac{\partial b'X'y}{\partial b} = X'y$

Note

More advanced matrix techniques can be found at this link

1.0.1 Special Matrices

Square Matrix

The number of rows (n) equals the number of columns (n) for the matrix. The following is an example of a square matrix:

$$\begin{bmatrix} 10 & 5 & 9 \\ 4 & 4 & 3 \\ 6 & 17 & 2 \end{bmatrix} \tag{10}$$

Symmetric Matrix

A symmetric matrix has the following property: $A^T = A$. This means that $a_{ij} = a_{ji}$ for all i, j. Notice that this implies that a symmetric matrix has to be a square matrix ($n \times n$). The following is an example of a symmetric matrix:

$$\begin{bmatrix} 1 & 5 & 6 \\ 5 & 4 & 7 \\ 6 & 7 & 2 \end{bmatrix} \tag{11}$$

Idempotent Matrix

An idempotent matrix (A) has the following property: AA = A

Identity Matrix

An $n \times n$ identity matrix (either donated as I or I_n) has 1's on the diagonal and 0's elsewhere. Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Multiplying any matrix by an identity matrix will return that matrix.

$$AI = A \tag{13}$$
$$IB = B \tag{14}$$

Nonsingular Matrix

Another name for an invertible matrix is a nonsingular matrix. A nonsingular matrix has a nonzero determinant.

Orthogonal Matrix

A square matrix, Q, is orthogonal if:

$$Q^T Q = QQ^T = I$$

Notice that this definition implies that $Q^T = Q^{-1}$.

Partition Matrix

A partitioned matrix is a matrix that is broken up into partitions (also called blocks).

$$\left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]$$
$$= \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

In order to perform certain operations we need to have our partitioned matrices partitioned appropriately.
Addition and subtraction: If we are adding $A + B$ or subtracting $A - B$, we need them to be the same size. Also, they need to be partition the same way.

$$\left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array}\right]$$
$$= \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$\left[\begin{array}{c|cc} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \hline b_{31} & b_{32} & b_{33} \end{array}\right]$$
$$= \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

Thus, $A + B$ will be defined as:

$$\left[\begin{array}{c|c} A_{11} + B_{11} & A_{12} + B_{12} \\ \hline A_{21} + B_{21} & A_{22} + B_{22} \end{array}\right]$$

Matrix Multiplication: We can also matrix multiply two partitioned matrices. Notice that if we are multiply AB, the number of columns in A has to be equal to the number of rows in B. If this is satisfied, We can used partitioned matrices and treat the submatrices as elements. If X and Y are $m \times n$ matrices, and after partitioning they are defined as:

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
$$Y = \begin{bmatrix} E \\ F \end{bmatrix}$$

Then it follows that XY is defined as:

$$XY = \begin{bmatrix} AE + BF \\ CE + DF \end{bmatrix}$$

Notice that this implies that the number of columns of A has to be equal to the number of rows in E and F, and also the number of columns in B has to be equal to the number of rows in E and F.

2 Linear Spaces

Recall that $\mathbb{R}$ is the set of all real numbers. $\mathbb{R}^n$ where $n \geq 1$ is a set that contains all n-tuples of real numbers. In other words, a vector in $\mathbb{R}^n$ would contain n elements that are in R.

Note

A set of the form $\mathbb{R}^n$ is often referred to as an Euclidean space.

Vector Space

A vector space is a collection of vectors which can either be added or scalar multiplied. A vector space is a non-empty set V that has the following properties (assuming $v, w, z \in V$):

1. $u + v \in V$
2. $cv \in V$
3. $u + v = v + u$
4. $(u + v) + w = u + (v + w)$
5. $a(bw) = (ab)w$ where $a, b \in \mathbb{R}$
6. $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$
7. For every $v \in V$, there exists a $w \in V$ such that $v + w = 0$
8. $Iv = v$
9. $c(v + w) = cv + cw$ for all $c \in \mathbb{R}$
10. $(k + c)u = ku + cu$ for all $k, c \in \mathbb{R}$

The vector space that we most commonly work with is $\mathbb{R}^n$.

Practice

Show that the set $\mathbb{R}^n$ is a vector space.

Subspace

A subset U of V is called a subspace of V if it is also a vector space. To check if U is a subspace, you only need to check that the following properties hold:

1. **Additive Identity:** $\mathbf{0} \in U$
2. **Closed under addition:** $u + v \in U$ if $u, v \in U$
3. **Closed under multiplication:** if $a \in \mathbb{R}$ and $u \in U$, then $au \in U$

When we get to the proof sections, we will look at different subsets, and you will be asked to show whether different subsets are subspaces.

Example

Is the subset $\{\mathbf{0}\}$ where $\mathbf{0} \in \mathbb{R}^n$ a subspace of $\mathbb{R}^n$?
We need to check that the 3 properties above hold:

1. $\mathbf{0} \in \{\mathbf{0}\}$
2. $\mathbf{0} + \mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$
3. $a\mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$

   Since the properties hold, $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

3 Exercises

1. Let A be a 3×3 matrix with $det(A) = 6$. Find each of the following if possible:

   (a) $\det(A^T)$

   (b) $\det(A + I)$

   (c) $\det(3A)$

   (d) $\det(A^4)$

2. A property of traces is that $tr(AB) = tr(BA)$. Using this property, show that $tr(ABC) = tr(CBA) = tr(ACB)$.

3. *This problem was taken from last year's problem set. It is such a great problem I felt that I needed to include it. Please do not look at last year's solution.*

   Let X be a $n \times k$ real matrix. Define projection matrix $P := X(X'X)^{-1}X'$ and orthogonal matrix $M := I_n - P$. (You can assume $(X'X)^{-1}$ exists.)

   (a) Show that P and M are symmetric and idempotent.

   (b) Show that $tr(P) = k$, $tr(M) = n - k$.

4. Let V be defined as follows:

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$$

   Surprise, surprise, V is not a vector space. Show by counterexample which properties (which are listed in the notes) are violated.