WSU Economics PhD Mathcamp Notes

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These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Proofs

1.1 Mathematical Induction

Let P(n) be a statement, where $n \in \mathbb{N}$. To prove by induction, we need to prove two things:

1. A base case (usual base case is $n = 1$)
2. The inductive step: $\forall k \in \mathbb{N}$, the implication: $P(k) \Rightarrow P(k+1)$ is true.

Example 1

Show that the sum of the first n positive integers is $n(n+1)/2$. Or in other words:

$$1 + 2 + 3 + 4 + ... + n = n(n+1)/2$$

Proof Let $P(n) : 1 + 2 + 3 + 4 + ... + n = n(n+1)/2$ where $n \in \mathbb{N}$

1. **Base case**: $P(1) : 1 = 1(1+1)/2 = 1$. Thus the base case is true.
2. **Inductive step**: Assume $P(k)$ is true for some $k \in \mathbb{N}$, thus:

$$P(k) : 1 + 2 + 3 + 4 + ... + k = k(k+1)/2$$

Now we show that $P(k+1)$ is true, or that $1+2+3+4+...+k+(k+1) = (k+1)(k+2)/2$
$1+2+3+4+...+k+(k+1) = k(k+1)/2+(k+1) = k(k+1)/2+2(k+1)/2 = (k+2)(k+1)/2$
By induction, $P(n)$ is true for every (postive) integer n.

Example 2

Show that for every positive n,

$$\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n+1)(n+2)}$$

Proof Let $P(n) : \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n+1)(n+2)} = \frac{n}{2n+4}$ where $n \in \mathbb{N}$

1. **Base case**: $P(1) : 1 = \frac{1}{2 \cdot 3} = \frac{1}{6}$. Thus the base case is true.

2. **Inductive step**: Assume $P(k)$ is true for some $k \in \mathbb{N}$, thus:

$$P(k) : \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(k+1)(k+2)}$$

Now we show that $P(k+1)$ is true, or that

$$\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+1)+4}$$

Now we show that $P(k+1)$ is true:

$$\begin{aligned}
\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} &= \frac{k}{2k+4} + \frac{1}{(k+2)(k+3)} \\
&= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)} \\
&= \frac{k(k+3)}{2(k+2)(k+3)} + \frac{2}{2(k+2)(k+3)} \\
&= \frac{k^2+3k+2}{2(k+2)(k+3)} \\
&= \frac{(k+1)(k+2)}{2(k+2)(k+3)} \\
&= \frac{(k+1)}{2(k+3)} \\
&= \frac{(k+1)}{2(k+1)+4}
\end{aligned}$$

By induction, $P(n)$ is true for every positive integer n.

Example 3

Show for every nonnegative integer n:

$$2^n > n$$

Proof Let $P(n) : 2^n > n$ where $n \in \mathbb{N} \cup \{0\}$

1. **Base case**: $P(1) : 2^0 > 0$. Thus the base case is true.

2. **Inductive step**: Assume $P(k)$ is true for some $k \in \mathbb{N} \cup \{0\}$, thus:

$$P(k) : 2^k > k$$

Now we show that $P(k+1)$ is true, or that $2^{k+1} > k+1$. Notice for $k \geq 1$:

$$
\begin{aligned}
2^k &> k \\
2 \cdot 2^k &> 2k \\
&= k + k \\
&\geq k + 1 \text{ since } k \geq 1
\end{aligned}
$$

Thus $2^{k+1} > k + 1$. By induction, $P(n)$ is true for every (positive) integer n.

Example 4

Show for sets $A_1, A_2, \ldots, A_n$ where $n \geq 2$, then:

$$\overline{A_1 \cup A_2 \cup \ldots \cup A_n} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n}$$

Proof

1. **Base case**: Notice that the base case, $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$
2. **Inductive step**: Assume $P(k)$ is true for some $k \geq 2$, thus:

   $$\overline{A_1 \cup A_2 \cup \ldots \cup A_k} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_k}$$

   Now we show that $P(k+1)$ is true, or that $k \geq 3$:

   $$\overline{A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup A_k \cup \overline{A_{k+1}}$$

   Let $T = A_1 \cup A_2 \cup \ldots \cup A_k$. Thus (given $P(k)$), we find that:

   $$\overline{T \cup A_{k+1}} = \overline{T} \cap \overline{A_{k+1}}$$

   Or in other words:

   $$\overline{A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup A_k \cup \overline{A_{k+1}}$$

   Thus, by induction, $P(n)$ is true for every $n \geq 2$.

2 Relations

A (binary) **relation**, R, from set A to set B is a subset of $A \times B$. Since R is a subset of $A \times B$, it is a set of ordered pairs. If $a \in A$ and $b \in B$, we say $(a, b) \in R$ if a is related to b. We can also write aRb if this holds. If an ordered pair $(c, d) \in A \times B$ is not in the relation R, then we could write either $(c, d) \notin R$ or $c \not R d$.

Example

If $A = \{t, u, v\}$ and $B = \{1, 2\}$, we see that:

$$A \times B = \{(t, 1), (t, 2), (u, 1), (u, 2), (v, 1), (v, 2)\}$$

An example of an relation R would be:

$$R = \{(t, 2), (u, 1), (u, 2)\}$$

Notice that $R \subseteq (A \times B)$

If R is the relation from A to B, then the domain of R is a subset of A defined by:

$$\text{dom}R = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$$

Likewise, the range is a subset of B defined by:

$$\text{ran}R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$$

The inverse of a relation R from A to B, is denoted R^{-1}, and is defined as:

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$$

Lastly, we can define a relation R from A to A. When we do so, we just call R a relation on A.

2.1 Properties of Relations

Below are some possible properties of a relation R on X:

1. R is **reflexive** $\Leftrightarrow xRx$ for any $x \in X$.
2. R is **transitive** $\Leftrightarrow (xRy \text{ and } yRz \Rightarrow xRz)$ for any $x, y, z \in X$
3. R is **symmetric** $\Leftrightarrow xRy$ and yRx for any $x, y \in X$
4. R is **complete** $\Leftrightarrow xRy$ or yRx for any $x, y \in X$
5. R is **antisymmetric** $\Leftrightarrow (xRy \text{ and } yRx \Leftrightarrow x = y)$ for any $x, y \in X$

Practice

Let $S = \{a, b, c\}$. Which of the properties reflexive, transitive, and symmetric do the relations below possess if the relations are from S to S?

1. $R_1 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a), (b, b), (c, c)\}$

2. $R_2 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a)\}$

3. $R_3 = \{(b, c), (c, b), (a, a), (b, b), (c, c)\}$

Relations can often be defined using set builder notation. Below is an example of a relation from $\mathbb{R}$ to $\mathbb{R}$:

$$R_4 = \{(a, b) \in \mathbb{R}^2 : a > b\}$$

Notice that $(2, 1), (\sqrt{5}, -3), (\pi, 0) \in R_4$ since $2 > 1, \sqrt{5} > -3,$ and $\pi > 0$.
However, $(2, 4), (-2, 3.4), (-3, \sqrt{5}) \notin R_4$ since $2 \not> 4, -2 \not> 3.4,$ and $-3 \not> \sqrt{5}$.

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onsider the relation R_4, namely $R_4 = \{(a, b) \in \mathbb{R}^2 : a > b\}$. Which properties reflexive, symmetric, and transitive does the relation R_4 possess?

- R_4 is not reflexive as $(a, a) \notin R_4$ since $a \not> a$ for any $a \in \mathbb{R}$.
- R_4 is not symmetric. Counterexample, let $a = 5$ and $b = 3$. Notice that $(5, 3) \in R_4$ but $(3, 5) \notin R_4$.
- R_4 is transitive as it holds that if $a > b$ and $b > c$, then $a > c$.

Practice

Consider $S \in \mathbb{R}$. Let the following be relations from S to S. Show that the following relations are reflexive, transitive, and symmetric. If a property does not hold, provide a counterexample to show that that property does not hold.

1. $R_5 = \{(a, b) \in S \times S : a \geq b\}$
2. $R_6 = \{(a, b) \in S \times S : a > b\}$
3. $R_7 = \{(a, b) \in S \times S : ab \geq 0\}$

Exercises

1. Let $X_1, X_2, \ldots, X_n$ be matrices where $n \in \mathbb{N}$. Using mathematical induction, show that $(X_1 X_2 \ldots X_n)^T = X_n^T \ldots X_2^T X_1^T$.

2. Using mathematical induction, show that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ where $n \in \mathbb{N}$.

3. A relation R is defined on $\mathbb{Z}$ by aRb if $|a-b| \leq 2$. Which of the properties reflexive, symmetric, and transitive does the relation R possess? Justify your answers.