WSU Economics PhD Mathcamp Notes

Joe Patten

July 31, 2019

These notes are to accompany Mathematics for Economists by Simon and Blume.

1 Real Analysis

1.1 Functions

A relation f from A to B is a **function**, which we write as $f : A \to B$, iff:

1. for every $a \in A$, there exists a $b \in B$
2. if $(a, b_1) \in f$ and $(a, b_2) \in f$, it must be the case that $b_1 = b_2$

If $(a, b) \in f$, we can write $f(a) = b$. b is called the image of a, and a is referred to as the preimage. When we write $f(a) = b$, we say that f maps a into b.

Example

Let $A = \{a, b, c\}$ and $B = \{3, 6, 7, 8\}$. f_1 and f_2 are an examples of functions:

$$f_1 = \{(a, 3), (b, 8), (c, 7)\}$$
$$f_2 = \{(a, 8), (b, 7), (c, 8)\}$$

f_3 and f_4 are examples of relations that are not functions:

$$f_3 = \{(a, 3), (a, 6), (b, 7), (c, 8)\}$$
$$f_4 = \{(b, 6), (c, 7)\}$$

A common function that you have seen before is the function $f(x) = x^2$. We can set f to be a set of all possible ordered pairs for $f(x) = x^2$:

$$f = \{(x, x^2) : x \in \mathbb{R}\}$$

1.2 Set of All Functions

Notice that we can write a number of functions from A and B. We denote the set of all functions from A to B by B^A. More formally, this set is defined as:

$$B^A = \{f : f : A \to B\}$$

1.3 One-to-One Functions

A function, f, from A to B is said to be **one-to-one** (or **injective**) if every two distinct values of A have distinct images in B. In other words, for every $a, a' \in A$, if $a \neq a'$, then $f(a) \neq f(a')$.

Example

Let A = {x,y,z} and B = {a,b,c,d }. f_1 and f_2 are examples of one-to-one functions from A to B:

$$f_1 = \{(x,b),(y,a),(z,d)\}$$
$$f_2 = \{(x,b),(y,c),(z,d)\}$$

f_3 and f_4 are examples of functions from A to B that are not one-to-one:

$$f_3 = \{(x,b),(y,b),(z,d)\}$$
$$f_4 = \{(x,d),(y,d),(z,d)\}$$

1.4 Onto Functions

A function, f, from A to B is said to be **onto** (or **surjective**) if every element of the codomain (in this case, B) is the image of some element of A.

Example

Let A = { e,f,g,h } and B = {1,2,3}. f_1 and f_2 are examples of onto functions from A to B:

$$f_1 = \{(e,1),(f,2),(g,3),(h,1)\}$$
$$f_2 = \{(e,3),(f,2),(g,2),(h,1)\}$$

f_3 and f_4 are examples of functions from A to B that are not onto:

$$f_3 = \{(e,1),(f,2),(g,2),(h,1)\}$$
$$f_4 = \{(e,1),(f,1),(g,1),(h,1)\}$$

1.5 Bijective Functions

A function, f, from A to B is said to be **bijective** (or a **one-to-one correspondence**) if it is one-to-one and onto.

1.6 Inverse Functions

Let $f : A \to B$ be a function. Then the **inverse** relation, f^{-1}, is a function from B to A iff f is bijective. Also if f is bijective, then f^{-1} is bijective.

1.7 Function Operations

Let f and g be functions mapping from $\mathbb{R}$ to $\mathbb{R}$. We can perform the following operations:

1. $(f + g)(x) = f(x) + g(x)$

2. $(fg)(x) = f(x) \cdot g(x)$
3. $(fg)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
4. $(g \circ f)(x) = g(f(x))$

Item 3 comes from the chain rule. Item 4 is called a composition.

1.8 Monotonic Functions

A function $f : A \to B$ is (weakly) increasing on A if $x < y \Rightarrow f(x) \leq f(y)$ and (weakly) decreasing when $x < y \Rightarrow f(x) \geq f(y)$. A function $f : A \to B$ is *strictly* increasing on A if $x < y \Rightarrow f(x) < f(y)$ and *strictly* decreasing when $x < y \Rightarrow f(x) > f(y)$. A function is said to be **monotonic** iff it is an increasing or decreasing function, and strictly monotonic iff it is strictly increasing or strictly decreasing.

Example

I will show that $f(x) = x^2 + 1$ is strictly increasing for all $x \in \mathbb{R}_+$, where $\mathbb{R}_+$ is defined as: $\mathbb{R}_+ = \{y \in \mathbb{R} : y \geq 0\}$

Solution: Take two arbitrary points, $x_1, x_2 \in \mathbb{R}_+$.
Assume without of generality that $0 \leq x_1 < x_2$.
Consider the difference of images:
$f(x_2) - f(x_1) = (x_2^2 + 1) - (x_1^2 + 1) = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$
Notice that $(x_2 - x_1)(x_2 + x_1) > 0$ since $x_1 \geq 0$ and $x_2 > 0$, and by assumption $x_2 > x_1$.
Thus, $f(x) = x^2 + 1$ is strictly increasing over the domain of $\mathbb{R}_+$

Practice

Show that the function $f(x) = log(x)$ is strictly increasing for all $x \in \mathbb{R}_{++}$, where $\mathbb{R}_{++}$ is defined as: $\mathbb{R}_{++} = \{y \in \mathbb{R} : y > 0\}$

When a strictly increasing function is applied to a set, we refer to this application as a (positive) **monotonic transformation**. The monotonically transformed set keeps it ordering, in other words, if $a, b \in S$ and $a > b$, and f is a strictly increasing function, then $f(a) > f(b)$.

2 Metric Spaces

Let X be a set. $d : X \times X \to \mathbb{R}$ is a valid metric or distance function iff:

1. $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$
2. $d(x, y) = d(y, x) \quad \forall x, y \in X$
3. $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

If d satisfies the above properties, then (X, d) is said to be a **metric space**.

Example

Let x, y be vectors in X. Some examples of common metrics are:

1. Absolute value metric

$$d_1(x, y) = |x - y|$$

Naturally, d_1 is defined on $\mathbb{R}$. If we were to consider a higher-dimensional space, then we can define the ℓ^1 metric:

$$d(x, y) = |x_1 - x_1| + |x_2 - x_2| + \ldots + |x_n - x_n|$$

2. Euclidean metric (also known as ℓ^2)

$$d_2(x, y) = ||x - y|| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$$

3. Square metric (also known as ℓ^∞)

$$d_3(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_k - y_k|\}$$

Practice

1. Show that $(\mathbb{R}, d_1)$ is a valid metric space.
2. Show that $(\mathbb{R}^2, d_2)$ is a valid metric space.
3. Show that $(\mathbb{R}^n, d_3)$ is a valid metric space.

Exercises

1. Verify that the following are valid metric spaces:
   (a) $(\mathbb{R}^2, \rho)$ where ρ is defined as:
   $$\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$
   (b) $(\mathbb{R}^n, \ell^1)$
2. Show that the following are strictly increasing functions (like we did in class):
   (a) $f(x) = e^x + 2x$
   (b) $f(x) = x^3 - x^2$ where $x \geq 1$
3. Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ (and a justification as to why) such that:
   (a) f is onto and one-to-one.
   (b) f is one-to-one but not onto.
   (c) f is onto but not one-to-one.