

# Real Analysis

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## 1 Relations and Functions

### 1.1 Relations

A (binary) **relation**,  $R$ , from set  $A$  to set  $B$  is a subset of  $A \times B$ . Since  $R$  is a subset of  $A \times B$ , it is a set of ordered pairs. If  $a \in A$  and  $b \in B$ , we say  $(a, b) \in R$  if  $a$  is related to  $b$ . We can also write  $aRb$  if this holds. If an ordered pair  $(c, d) \in A \times B$  is not in the relation  $R$ , then we could write either  $(c, d) \notin R$  or  $c \not R d$ .

#### Example

If  $A = \{t, u, v\}$  and  $B = \{1, 2\}$ , we see that:

$$A \times B = \{(t, 1), (t, 2), (u, 1), (u, 2), (v, 1), (v, 2)\}$$

An example of an relation  $R$  would be:

$$R = \{(t, 2), (u, 1), (u, 2)\}$$

Notice that  $R \subseteq (A \times B)$

If  $R$  is the relation from  $A$  to  $B$ , then the domain of  $R$  is a subset of  $A$  defined by:

$$\text{dom}R = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$$

Likewise, the range is a subset of  $B$  defined by:

$$\text{ran}R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}$$

The inverse of a relation  $R$  from  $A$  to  $B$ , is denoted  $R^{-1}$ , and is defined as:

$$R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$$

Lastly, we can define a relation  $R$  from  $A$  to  $A$ . When we do so, we just call  $R$  a relation on  $A$ .

### 1.2 Properties of Relations

Below are some possible properties of a relation  $R$  on  $X$ :

1.  $R$  is **reflexive**  $\Leftrightarrow xRx$  for any  $x \in X$ .
2.  $R$  is **transitive**  $\Leftrightarrow (xRy \text{ and } yRz \Rightarrow xRz)$  for any  $x, y, z \in X$
3.  $R$  is **symmetric**  $\Leftrightarrow xRy \text{ and } yRx$  for any  $x, y \in X$
4.  $R$  is **complete**  $\Leftrightarrow xRy \text{ or } yRx$  for any  $x, y \in X$
5.  $R$  is **antisymmetric**  $\Leftrightarrow (xRy \text{ and } yRx \Leftrightarrow x = y)$  for any  $x, y \in X$

### Practice

Let  $S = \{a, b, c\}$ . Which of the properties reflexive, transitive, and symmetric do the relations below possess if the relations are from  $S$  to  $S$ ?

1.  $R_1 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a), (b, b), (c, c)\}$
2.  $R_2 = \{(a, c), (c, a), (a, b), (b, a), (b, c), (c, b), (a, a)\}$
3.  $R_3 = \{(b, c), (c, b), (a, a), (b, b), (c, c)\}$

Relations can often be defined using set builder notation. Below is an example of a relation from  $\mathbb{R}$  to  $\mathbb{R}$ :

$$R_4 = \{(a, b) \in \mathbb{R} : a > b\}$$

Notice that  $(2, 1), (\sqrt{5}, -3), (\pi, 0) \in R_4$  since  $2 > 1, \sqrt{5} > -3,$  and  $\pi > 0$ . However,  $(2, 4), (-2, 3.4), (-3, \sqrt{5}) \notin R_4$  since  $2 \not> 4, -2 \not> 3.4,$  and  $-3 \not> \sqrt{5}$ .

### Practice

Consider  $S \in \mathbb{R}$ . Let the following be relations from  $S$  to  $S$ . Show that the following relations are reflexive, transitive, and symmetric. If a property does not hold, provide a counterexample to show that that property does not hold.

1.  $R_5 = \{(a, b) \in S \times S : a \geq b\}$
2.  $R_6 = \{(a, b) \in S \times S : a > b\}$
3.  $R_7 = \{(a, b) \in S \times S : ab \geq 0\}$

## 1.3 Functions

A relation  $f$  from  $A$  to  $B$  is a **function**, which we write as  $f : A \rightarrow B$ , iff:

1. for every  $a \in A$ , there exists a  $b \in B$
2. if  $(a, b_1) \in f$  and  $(a, b_2) \in f$ , it must be the case that  $b_1 = b_2$

If  $(a, b) \in f$ , we can write  $f(a) = b$ .  $b$  is called the image of  $a$ , and  $a$  is referred to as the preimage. When we write  $f(a) = b$ , we say that  $f$  maps  $a$  into  $b$ .

### Example

Let  $A = \{a, b, c\}$  and  $B = \{3, 6, 7, 8\}$ .  $f_1$  and  $f_2$  are an examples of functions:

$$f_1 = \{(a, 3), (b, 8), (c, 7)\}$$

$$f_2 = \{(a, 8), (b, 7), (c, 8)\}$$

$f_3$  and  $f_4$  are examples of relations that are not functions:

$$f_3 = \{(a, 3), (a, 6), (b, 7), (c, 8)\}$$

$$f_4 = \{(b, 6), (c, 7)\}$$

a common function that you have seen before is the function  $f(x) = x^2$ . We can set  $f$  to be a set of all possible ordered pairs for  $f(x) = x^2$ :

$$f = \{(x, x^2) : x \in \mathbb{R}\}$$

## 1.4 Set of All Functions

Notice that we can write a number of functions from  $A$  and  $B$ . We denote the set of all functions from  $A$  to  $B$  by  $B^A$ . More formally, this set is defined as:

$$B^A = \{f : f : A \rightarrow B\}$$

## 1.5 One-to-One Functions

A function,  $f$ , from  $A$  to  $B$  is said to be **one-to-one** (or **injective**) if every two distinct values of  $A$  have distinct images in  $B$ . In other words, for every  $a, a' \in A$ , if  $a \neq a'$ , then  $f(a) \neq f(a')$ .

### Example

Let  $A = \{x, y, z\}$  and  $B = \{a, b, c, d\}$ .  $f_1$  and  $f_2$  are examples of one-to-one functions from  $A$  to  $B$ :

$$f_1 = \{(x, b), (y, a), (z, d)\}$$

$$f_2 = \{(x, b), (y, c), (z, d)\}$$

$f_3$  and  $f_4$  are examples of functions from  $A$  to  $B$  that are not one-to-one:

$$f_3 = \{(x, b), (y, b), (z, d)\}$$

$$f_4 = \{(x, d), (y, d), (z, d)\}$$

## 1.6 Onto Functions

A function,  $f$ , from  $A$  to  $B$  is said to be **onto** (or **surjective**) if every element of the codomain (in this case,  $B$ ) is the image of some element of  $A$ .

### Example

Let  $A = \{e, f, g, h\}$  and  $B = \{1, 2, 3\}$ .  $f_1$  and  $f_2$  are examples of onto functions from  $A$  to  $B$ :

$$f_1 = \{(e, 1), (f, 2), (g, 3), (h, 1)\}$$

$$f_2 = \{(e, 3), (f, 2), (g, 2), (h, 1)\}$$

$f_3$  and  $f_4$  are examples of functions from  $A$  to  $B$  that are not onto:

$$f_3 = \{(e, 1), (f, 2), (g, 2), (h, 1)\}$$

$$f_4 = \{(e, 1), (f, 1), (g, 1), (h, 1)\}$$

## 1.7 Bijective Functions

A function,  $f$ , from  $A$  to  $B$  is said to be **bijective** (or a **one-to-one correspondence**) if it is one-to-one and onto.

## 1.8 Inverse Functions

Let  $f : A \rightarrow B$  be a function. Then the **inverse** relation,  $f^{-1}$ , is a function from  $B$  to  $A$  iff  $f$  is bijective. Also if  $f$  is bijective, then  $f^{-1}$  is bijective.

## 1.9 Function Operations

Let  $f$  and  $g$  be functions mapping from  $\mathbb{R}$  to  $\mathbb{R}$ . We can perform the following operations:

1.  $(f + g)(x) = f(x) + g(x)$

2.  $(fg)(x) = f(x) \cdot g(x)$

3.  $(fg)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

4.  $(g \circ f)(x) = g(f(x))$

Item 3 comes from the chain rule. Item 4 is called a composition.

## 1.10 Monotonic Functions

A function  $f : A \rightarrow B$  is (weakly) increasing on  $A$  if  $x < y \Rightarrow f(x) \leq f(y)$  and (weakly) decreasing when  $x < y \Rightarrow f(x) \geq f(y)$ . A function  $f : A \rightarrow B$  is *strictly* increasing on  $A$  if  $x < y \Rightarrow f(x) < f(y)$  and *strictly* decreasing when  $x < y \Rightarrow f(x) > f(y)$ . A function is said to be **monotonic** iff it is an increasing or decreasing function, and strictly monotonic iff it is strictly increasing or strictly decreasing.

### Example

I will show that  $f(x) = x^2 + 1$  is strictly increasing for all  $x \in \mathbb{R}_+$ , where  $\mathbb{R}_+$  is defined as:  
 $\mathbb{R}_+ = \{y \in \mathbb{R} : y \geq 0\}$

*Solution:* Take two arbitrary points,  $x_1, x_2 \in \mathbb{R}_+$ .

Assume without of generality that  $0 \leq x_1 < x_2$ .

Consider the difference of images:

$$f(x_2) - f(x_1) = (x_2^2 + 1) - (x_1^2 + 1) = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1)$$

Notice that  $(x_2 - x_1)(x_2 + x_1) > 0$  since  $x_1 \geq 0$  and  $x_2 > 0$ , and by assumption  $x_2 > x_1$ .

Thus,  $f(x) = x^2 + 1$  is strictly increasing over the domain of  $\mathbb{R}_+$

### Practice

Show that the function  $f(x) = \log(x)$  is strictly increasing for all  $x \in \mathbb{R}_{++}$ , where  $\mathbb{R}_{++}$  is defined as:  $\mathbb{R}_{++} = \{y \in \mathbb{R} : y > 0\}$

When a strictly increasing function is applied to a set, we refer to this application as a (positive) **monotonic transformation**. The monotonically transformed set keeps its ordering, in other words, if  $a, b \in S$  and  $a > b$ , and  $f$  is a strictly increasing function, then  $f(a) > f(b)$ .

## 2 Metric Spaces

Let  $X$  be a set.  $d : X \times X \rightarrow \mathbb{R}$  is a valid metric or distance function iff:

1.  $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in X$
2.  $d(x, y) = d(y, x) \quad \forall x, y \in X$
3.  $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

If  $d$  satisfies the above properties, then  $(X, d)$  is said to be a **metric space**.

### Example

Let  $x, y$  be vectors in  $X$ . Some examples of common metrics are:

1. Absolute value metric

$$d_1(x, y) = |x - y|$$

2. Euclidean metric

$$d_2(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^k (x_i - y_i)^2}$$

3. Square metric

$$d_3(x, y) = \max\{|x_1 - y_1|, \dots, |x_k - y_k|\}$$

### Practice

1. Show that  $(\mathbb{R}, d_1)$  is a valid metric space.
2. Show that  $(\mathbb{R}^2, d_2)$  is a valid metric space.
3. Show that  $(\mathbb{R}^n, d_3)$  is a valid metric space.

### 3 More Set Theory

Consider the metric space  $(\mathbb{R}, d_1)$ . A set  $S \subseteq \mathbb{R}$  is **bounded above** if there exists  $b \in \mathbb{R}$  such that  $a \leq b$  for all  $a \in A$ . We call  $b$  an **upper bound** of  $S$ . A set  $S \subseteq \mathbb{R}$  is **bounded below** if there exists  $c \in \mathbb{R}$  such that  $a \geq c$  for all  $c \in A$ . We call  $C$  a **lower bound** of  $S$ .

The **supremum**  $s$ , or least upper bound, for a set  $S \subseteq \mathbb{R}$  if:

1.  $s$  is an upper bound of  $S$ .
2. if  $d$  is any upper bound of  $S$ , then  $s \leq d$ .

The **infimum**, or greatest lower bound, for a set  $S \subseteq \mathbb{R}$  is defined similarly.

Consider the metric space  $(\mathbb{R}, d_1)$ . The number  $m$  is the **maximum** of a set  $S$  if  $m \in S$ , and  $m \geq a$  for all  $a \in S$ . The **minimum** of a set can be defined similarly.

#### Practice

Determine the supremum and infimum for each of the following sets in  $\mathbb{R}$ . Also determine if the supremum and infimum are equal to the maximum and minimum respectively of each set:

1.  $[0, 2]$
2.  $(0, 2)$
3.  $[0, 2] \cup \{3\}$

## 4 Sequences

A **sequence** is a function  $x : \mathbb{N} \rightarrow X$ , which is either written as  $(x_n)$  or  $\{x_n\}$ . In other words, to define a sequence, we take numbers from the set of natural numbers (starting from 1 and counting up), and using a function, assign them to elements from a set  $X$ . Since a sequence is a function, we can write the  $n^{\text{th}}$  element of a sequence as  $x(n)$ , however, it is more commonly written as  $x_n$ . Recall that order does not matter for sets. This is not the case with sequences. For example,  $(1, 1, 2, \dots) \neq (1, 2, 1, \dots)$ . You will also notice that elements in a set can be repeated.

### 4.1 Sequence Convergence

Let  $(X, d)$  be a metric space. A sequence  $(x_n)$  **converges** to  $x$  if for every  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that whenever  $n > N$ , it follows that  $d(x_n, x) < \epsilon$ .

If the sequence  $(x_n)$  converges to  $x$ , then we can either write  $\lim x_n = x$  or  $(x_n) \rightarrow x$ .

Consider the sequence  $(x_n)$ , where  $x_n = \frac{1}{\sqrt{n}}$ . Find what it converges to, and show via proof that it converges to that value.

#### Proof

Let  $\epsilon$  be an arbitrary positive number. Choose a natural number such that  $N > \frac{1}{\epsilon^2}$ . Now, verify that this choice of  $N$  has the desired property. Let  $n \geq N$ . Then:

$$n > \frac{1}{\epsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \epsilon$$

Thus  $|x_n - 0| < \epsilon$

Template for  $(x_n) \rightarrow x$  proof in metric space  $(\mathbb{R}, d_1)$ :

- "Let  $\epsilon$  be arbitrary."
- Choose an  $N \in \mathbb{N}$  (this may take some work).
- Show that your choice of  $N$  works.
- "Assume  $n \geq N$ ."
- Now derive the inequality  $|x_n - x| < \epsilon$ .

A sequence converges to at most one limit.

#### Proof

We want to show that a sequence converges to at most one limit. I will show that this is the case when  $(\mathbb{R}, d_1)$  is the metric space.

*Proof:* Suppose to the contrary that a sequence converges to both  $x$  and  $x'$  where  $x \neq x'$ .

Since  $x \neq x'$ , we know that  $d(x, x') > 0$ .

Let  $\epsilon = d(x, x')/2$ .

By definition of convergence, we see that since  $x_n \rightarrow x$ ,  $\exists N_1$  s.t.  $d(x_n, x) < \epsilon$ .

By definition of convergence, we see that since  $x_n \rightarrow x'$ ,  $\exists N_2$  s.t.  $d(x_n, x') < \epsilon$ .

Now, let  $N = \max\{N_1, N_2\}$ . If  $n > N$ , then  $|x_n - x| < \epsilon$  and  $|x_n - x'| < \epsilon$ .

By the triangle inequality:  $|x - x'| \leq |x_n - x| + |x' - x_n| < |x - x'|$ .

But, this is a contradiction since  $|x - x'| < |x - x'|$ .

A sequence is said to be bounded if  $\exists M > 0$  s.t.  $|x_n| < M \quad \forall n \in \mathbb{N}$ .

### Practice

Show (via proof) that:

1.  $\lim \frac{2}{\sqrt{2n+4}} = 0$
2.  $\lim \frac{4n+1}{2n+4} = 2$

## 4.2 Algebraic Limit Theorem

Let  $\lim x_n = x$  and  $\lim y_n = y$ . Then:

1.  $\lim(cx_n)$
2.  $\lim(x_n + y_n) = x + y$
3.  $\lim(x_n y_n) = xy$
4.  $\lim(x_n/y_n) = x/y$ , given  $y \neq 0$ .

## 4.3 Squeeze Theorem

If  $x_n \leq y_n \leq z_n$  for all  $n \in \mathbb{N}$ , and  $\lim x_n = l$ ,  $\lim z_n = l$ , then  $\lim y_n = l$ .

## 4.4 Subsequence

A subsequence is derived from a sequence  $(x_n)$  by only keeping a subset of the elements while keeping the order of the sequence.

### Example

A subsequence of the sequence  $(1, 2, 3, 4, 5)$  is  $(1, 4, 5)$ .

## 4.5 Cauchy Criterion

A sequence is a **Cauchy sequence** if for every  $\varepsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that whenever  $m, n \geq N$  it follows that  $|x_n - x_m| < \varepsilon$ .

Every convergent sequence is a Cauchy sequence. The converse (i.e. every Cauchy sequence is convergent) is only true for certain metric spaces.

### Practice

Consider the metric space  $(\mathbb{R}, d_1)$ . Show that every convergent sequence is a Cauchy sequence.

A metric space  $(X, d)$  is complete if every Cauchy sequence in  $X$  converges to an element of  $X$ .

## 4.6 Monotonic Sequences

A sequence is increasing if  $x_n \leq x_{n+1}$  for all  $n \in \mathbb{N}$ , and is decreasing if  $x_n \geq x_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence is **monotonic** if it is either increasing or decreasing.



## 5 Topology

### 5.1 Open Sets

An open ball of radius  $\varepsilon > 0$  centered about point  $x \in X$  can be defined by:

$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$$

If we are working in the metric space  $(\mathbb{R}, d_1)$ , then the open ball is just an open interval, and is usually referred to as an  $\varepsilon$ -neighborhood. In the metric space  $(\mathbb{R}^2, d_2)$ , the open ball is a circle, and in the metric space  $(\mathbb{R}^3, d_2)$ , the open ball is a sphere.

A set  $A \subseteq X$  is open if for all points  $a \in A$  if there exists an open ball  $B_\varepsilon(a) \subseteq A$ .

#### Practice

Using the definition above and assuming the metric space is  $(\mathbb{R}, d_1)$ :

1. Show that  $B_\varepsilon(a)$  is an open set.
2. Show that  $\mathbb{R}$  is an open set.
3. Show that  $(0, 1)$  is an open set.

The following theorems hold for open sets:

1. The union of open sets is open.
2. The finite intersection of open sets is open.

### 5.2 Closed Sets

Let  $(X, d)$  be a metric space, and  $S$  be a subset of  $X$ . A point  $x$  is a **limit point** of set  $S$  iff  $\{B_\varepsilon(x) - \{x\}\} \cap S \neq \emptyset$  where  $\varepsilon > 0$ . For  $x$  to be a limit point of  $S$ , if we take an open ball around it, no matter what the size of that open ball, there will exist other points from  $S$  other than  $x$  in that open ball. Note: for a point  $x$  to be a limit point of  $S$ , it doesn't necessarily have to be an element of  $S$ . It only has to contain an element from  $S$  in its open ball for every  $\varepsilon > 0$ . The set of all limit points of a set  $S$  is usually denoted as  $S'$ .

We say that a set  $S$  is **closed** iff  $S$  contains all of its limit points. In other words,  $S' \subseteq S$ . It does not have to be the case that  $S = S'$ .

#### Practice

The set of natural numbers,  $\mathbb{N}$ , can be written in the form:  $\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \dots$  where  $\{n\}$  is said to be an isolated point. Is  $\{n\}$  a limit point? What does that tell us about the set  $\mathbb{N}$ , is it open, closed, or neither.

The following theorems hold for closed sets:

1. The finite union of closed sets is closed.
2. The intersection of closed sets is closed.

### 5.3 Open and Closed Sets

The following theorem is useful for determining if a set is open or closed: The complement of a closed set is open, and the complement of an open set is closed.

### Practice

1. Show that the empty set,  $\emptyset$ , is both closed and open.
2. Determine if  $[0, 1] \cup \{2\}$  is open, closed, or neither.

## 5.4 Compact Sets

A subset  $S$  in a Euclidean space is said to be compact iff  $S$  is closed and bounded.

A subset  $S$  is compact if every sequence in  $S$  has a subsequence that converges to a point in  $S$ .

### Practice

Show that for a compact set  $S \subseteq \mathbb{R}$ , the supremum and infimum of  $S$  are elements of  $S$ .

## 6 Advanced Theorems

### 6.1 Continuous Functions

A function  $f : A \rightarrow \mathbb{R}$  is **continuous** at  $c \in A$  if for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that when  $|x - c| < \delta$ , it follows that  $|f(x) - f(c)| < \varepsilon$ .  $f$  is said to be continuous on  $A$  if  $f$  is continuous at every point in the domain  $A$ .

Additional properties: Let  $f : A \rightarrow \mathbb{R}$  and  $g : A \rightarrow \mathbb{R}$  be continuous at a point  $c \in A$ . Then:

1.  $kf(x)$  is continuous at  $c$  for every  $k \in \mathbb{R}$
2.  $f(x) + g(x)$  is continuous at  $c$ .
3.  $f(x) \cdot g(x)$  is continuous at  $c$ .
4.  $\frac{f(x)}{g(x)}$  is continuous at  $c$ , given  $\frac{f(x)}{g(x)}$  exists.

### 6.2 Intermediate Value Theorem

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$ , and  $r$  is a real number such that  $f(a) \leq r \leq f(b)$  or  $f(z) \geq r \geq f(b)$ , then there exists a  $c \in [a, b]$  such that  $f(c) = r$ .

### 6.3 Fixed Points

Let  $f : X \rightarrow X$ . A point  $x^*$  is a **fixed point** of  $f$  iff  $f(x^*) = x^*$ . Notice that if we apply the function  $f$  on a fixed point  $x^*$  multiple times, we still get  $x^*$  as an output.

### 6.4 Brouwer's Fixed Point Theorem

Let  $(\mathbb{R}^n, d)$  be a metric space. If  $f : X \rightarrow X$  be a continuous function, where  $X$  is a compact and convex subset of  $\mathbb{R}^n$ , then there exists an  $x^* \in X$  such that  $f(x^*) = x^*$ .

#### Practice

Using the Intermediate Value Theorem, prove the Brouwer's Fixed Point Theorem in the metric space  $(\mathbb{R}, d_1)$

### 6.5 Contraction

We will use the notation  $f^n(x)$  to mean apply the function  $n$  times on  $x$ . In other words, if  $n = 3$ , then  $f^3(x) = f(f(f(x)))$ .

Let  $(X, d)$  be a metric space. A function  $f : X \rightarrow X$  is said to be a contraction iff there exists a  $\lambda < 1$  such that:

$$d(f(x), f(x')) \leq \lambda \cdot d(x, x')$$

for any  $x, x' \in X$ .

Notice that when a function is a contraction, when we apply the function to two points, the distance between the images are closer than the distance between the preimages.

#### Example

Let  $([\frac{1}{2}, 10], d_1)$  be a metric space, and let  $f$  be a function from the set  $[\frac{1}{2}, 10]$  into itself. We see that the function as  $f(x) = \sqrt{x}$  on this metric space is a contraction. Notice that no matter what  $x$  we start with (where  $x \in [\frac{1}{2}, 10]$ ),  $\lim_{n \rightarrow \infty} f^n(x) = 1$ .

Starting with  $x = \frac{1}{2}$ :

$$f\left(\frac{1}{2}\right) = 0.707107$$

$$f^2\left(\frac{1}{2}\right) = 0.84090$$

$$f^3\left(\frac{1}{2}\right) = 0.917004$$

⋮

$$f^{20}\left(\frac{1}{2}\right) = 0.999999$$

## 6.6 Contraction Mapping Theorem

Let  $(X, d)$  be a complete metric space, and  $f : X \rightarrow X$  be contraction. It follows that there exists a fixed point  $x^*$  of  $f$ , and for any  $x \in X$ ,  $\lim_{n \rightarrow \infty} f^n(x) = x^*$ .