PhD Mathcamp Final **Name:** ____________________

Summer 2019

Time Limit: 120 Minutes

Instructions: Some questions on this test may be a bit difficult. Relax, and answer all questions to the best of your ability (check every page to make sure you have answered everything). Note that partial solutions will receive partial credit, so putting something for a question will be better than leaving that question blank.

1. **(5 points)** Sets

   (a) Express the following set using set-builder notation $\{f(x) \in \mathbb{Z} : p(x)\}$, where $f(x)$ is a function of x, and $p(x)$ is a statement or condition of x.

   $$\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...\}$$

   $$\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}$$

   (b) Consider the metric space: $(\mathbb{R}, |\cdot|)$. is $(\frac{1}{2}, 1]$ open, closed, or neither? Justify your answer.

   The interval $(\frac{1}{2}, 1]$ is not open since if you put an open ball around 1, that ball contains elements not in $(\frac{1}{2}, 1]$. The interval $(\frac{1}{2}, 1]$ is not closed as it does not contain 0, which is a limit point of the interval.

   Consider the metric space: $([0, 1], |\cdot|)$. is $(\frac{1}{2}, 1]$ open, closed, or neither? Justify your answer.

   The interval $(\frac{1}{2}, 1]$ is open as there exists an $\varepsilon > 0$ such that if you put an open ball around any point, that open ball will be contained in $(\frac{1}{2}, 1]$

2. **(15 points)** Consider the following consumer utility max problem (and assume $m \geq p \geq 1$):

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \ln x_1 + x_2$$

$$\text{such that}$$
$$x_1 + px_2 \leq m$$
$$x_1 \geq 1$$

[1]

(a) Using the definition of strictly increasing function (without using derivatives), show that $f(x) = \ln x$ and $g(x) = x$ are strictly increasing functions.
Let $x_2 > x_1$. Notice:

$$f(x_2) - f(x_1) = \ln(x_2) - \ln(x_1) = \ln\left(\frac{x_2}{x_1}\right) > 0$$

$$g(x_2) - g(x_1) = x_2 - x_1 > 0$$

Therefore f and g are strictly increasing functions.

(b) Calculate the Hessian, $D^2 f_{(x_1,x_2)}$, of the objective function, $f(x_1, x_2) = \ln x_1 + x_2$, and show that it is negative semi-definite over the domain $x_1 \geq 1, x_2 \geq 0$.

$$Df_{(x_1,x_2)} = \begin{bmatrix} \frac{1}{x_1} & 1 \end{bmatrix}$$
$$D^2 f_{(x_1,x_2)} = \begin{bmatrix} -\frac{1}{x_1^2} & 0 \\ 0 & 0 \end{bmatrix}$$

Notice that for $x_1 \geq 1$, it follows that $-\frac{1}{x_1^2} < 0$. The first order principal minors ($-\frac{1}{x_1^2}$ and 0) are nonpositive, and the second order principal minor (0) is nonnegative. Thus the Hessian is negative semi-definite. This tells us that the objective function is concave.

[1] Recall that $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$

(c) Define the Lagrangian and find the Karush-Kuhn-Tucker conditions (you don't have to include the nonnegativity constraint of $x_2 \geq 0$).

$$L = \ln x_1 + x_2 - \lambda(x_1 + px_2 - m) - \mu(1 - x_1)$$

<u>KKT conditions</u>

$$\frac{\partial L}{\partial x_1} : \frac{1}{x_1^*} - \lambda^* + \mu^* = 0 \tag{1}$$

$$\frac{\partial L}{\partial x_2} : 1 - \lambda^* p = 0 \tag{2}$$

$$\lambda^*(x_1^* + px_2^* - m) = 0 \tag{3}$$

$$\mu^* x_1^* = 0 \tag{4}$$

$$\lambda^*, \mu^* \geq 0 \tag{5}$$

$$x_1^* + px_2^* \leq m \tag{6}$$

$$x_1^* \geq 1 \tag{7}$$

(d) In part (a) we showed that the objective function is strictly increasing in both inputs (x_1 and x_2). That means that the budget constraint ($x_1 + px_2 \leq m$) is binding, or in other words holds with equality ($x_1 + px_2 = m$). Give some intuition why this is the case.

The objective function (utility function) is strictly increasing in both inputs, meaning that the consumer will gain more utility by having more of either good. Thus, the consumer will spend all of his income (as he is better off buying x_1 and x_2).

(e) Assume that the constraints $x_1 \geq 1$ and $x_2 \geq 0$ are not binding. Using the conditions in (c) and (d), find the maximizers, x_1^* and x_2^*.

From equation (2), we find that $\lambda^* = \frac{1}{p}$, and from equation (1), we find that $x_1^* = \frac{1}{\lambda}$, thus $x_1^* = p$. Plugging this into the budget constraint, we find that $x_2^* = \frac{m}{p} - 1$.

I forgot to include the part "Assume that the constraints $x_1 \geq 1$ and $x_2 \geq 0$ are not binding" on the test. If however, we don't know that $x_1 \geq 1$ and $x_2 \geq 0$ are not binding, and the Lagrangian is:

$$L = \ln x_1 + x_2 - \lambda(x_1 + px_2 - m) - \mu_1(1 - x_1) + \mu_2 x_2$$

The FOCs then become:

$$\frac{\partial f}{\partial x_1} : \frac{1}{x_1} - \lambda + \mu_1 = 0$$
$$\frac{\partial f}{\partial x_2} : 1 - p\lambda + \mu_2 = 0$$

Notice that since the objective function is strictly increasing in its inputs, then at least one of μ_1 and μ_2 is positive. Then we have 1 of three cases:

<u>Case 1:</u> $\mu_1 = 0$ and $\mu_2 > 0$

$$\Rightarrow x_1^* = \frac{1}{\lambda^*}$$
$$\Rightarrow \lambda^* = \frac{1 + \mu_2^*}{p}$$
$$\Rightarrow x_1^* = \frac{p}{1 + \mu_2^*}$$

From Budget Constraint:

$$\Rightarrow x_2^* = \frac{m}{p} - \frac{1}{1 + \mu_2^*}$$

Since we know that $x_2^* = 0$:

$$\frac{m}{p} = \frac{1}{1 + \mu_2^*}$$
$$m = \frac{p}{1 + \mu_2^*}$$

Thus $x_1 = m$ and $x_2 = 0$.

<u>Case 2:</u> $\mu_1 > 0$ and $\mu_2 = 0$

$$\lambda^* = \frac{1}{p}$$
$$\Rightarrow x_1^* = \frac{1}{\frac{1}{p} - \mu_1^*}$$
$$\Rightarrow 1 = \frac{1}{\frac{1}{p} - \mu_1^*}$$
$$\Rightarrow x_2^* = \frac{m - 1}{p}$$

Thus, $x_1 = 0$, and $x_2^* = \frac{m-1}{p}$

Case 3: $\mu_1 = 0$ and $\mu_2 = 0$ Then we get the following from the FOCs and the budget constraint:

$$x_1^* = p$$
$$\Rightarrow x_2^* = \frac{m}{p} - 1$$

Notice that since $m \geq p \geq 1$, $x_1^* = p$ and $x_2^* = \frac{m}{p} - 1$ gives the maximum amount of utility.

3. **(10 points)** Prove the following:

   (a) Recall that the epigraph of a function from $\mathbb{R} \to \mathbb{R}$ is the set of points lying on or above the graph: *epi* $f = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, y \geq f(x)\}$. Show that if the epigraph of a function is convex, then the function is convex.

   Proof: We need to show that f is convex. In other words, for for $x_1, x_2 \in \mathbb{R}$, it follows that:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

   where $\lambda \in [0, 1]$

   Assume that *epi* f is convex. We will consider two elements that are in the graph of f, and are thus in the epigraph of f. Thus for $z_1, z_2 \in$ *epi* f where $z_1 = (x_1, f(x_1))$ and $z_2 = (x_2, f(x_2))$, it follows that $\lambda z_1 + (1 - \lambda)z_2 \in$ *epi* f. Let $(x_\lambda, y_\lambda) = z_\lambda = \lambda z_1 + (1 - \lambda)z_2$. Since z_λ is in the epigraph, by definition $y_\lambda \geq f(x_\lambda)$. This leads us to $f(\lambda x_1+(1-\lambda)x_2) \leq \lambda f(x_1)+(1-\lambda)f(x_2)$, which is the definition of f being convex.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function. Using the definition of concave function, show that

$$\frac{f(y) - f(x)}{y - x} \leq f'(x)$$

Hint: You might find the definition of derivative helpful: $f'(x) = \frac{f(x+h)-f(x)}{h}$ *where* h *is some small value (we take* $h \to 0$ *to find the derivative)*

We need the condition $y > x$ to hold in order for the inequality $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ to hold.

From the definition of concavity, we see that for $x, y \in f$, and $\lambda \in [0, 1]$, then:

$$f(\lambda y + (1 - \lambda x)) \geq \lambda f(y) + (1-\lambda)f(x)$$

$$\Rightarrow f(\lambda y + (1-\lambda)x) - f(x) \geq \lambda(f(y) - f(x)) \tag{8}$$

$$\Rightarrow \frac{f(\lambda y + (1-\lambda)x) - f(x)}{\lambda} \geq (f(y) - f(x)) \tag{9}$$

Notice that we need to introduce h into our equation. We need the following equation to hold:

$$x + h = \lambda y + (1 - \lambda x)$$

$$h = \lambda(y - x) \tag{10}$$

Using 10 and 9, we get:

$$\frac{f(x+h) - f(x)}{h}(y - x) \geq f(y) - f(x)$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} \geq \frac{f(y) - f(x)}{y - x}$$

$$\Rightarrow f'(x) \geq \frac{f(y) - f(x)}{y - x}$$

4. **(5 points)** Show that a convergent sequence in the metric space $(\mathbb{R}, |\cdot|)$ where $|\cdot|$ is the absolute value metric is a Cauchy sequence.

Let $\{x_n\}$ be a convergent sequence. Thus, for $\varepsilon_c > 0$ there exists an N such that $N \in \mathbb{N}$. Thus for $m, n \geq N$, it follows that:

$$|x_n - x| < \varepsilon_c \text{ and } |x_m - x| < \varepsilon_c$$

Notice:

$$|x_n - x| + |x_m - x| < 2\varepsilon_c$$

and

$$\begin{aligned} |x_n - x| + |x_m - x| &\geq |(x_n - x) - (x_m - x)| \\ &= |x_n - x_m| \end{aligned}$$

Let $2\varepsilon_c = \varepsilon$, then:

$$|x_n - x_m| < \varepsilon$$

Thus $\{x_n\}$ is Cauchy.

5. **(5 points)** Consider the metric space $(\mathbb{R}, |\cdot|)$ where $|\cdot|$ is the absolute value metric. The Brouwer's Fixed Point Theorem is as follows:

   Suppose that $X \subset \mathbb{R}$ is a nonempty, compact, convex set, and that $f : X \to X$ is a continuous function from X into itself. Then f has a fixed point; that is an $x^* \in X$ such that $x^* = f(x)$.

   Use the intermediate value theorem to prove Brouwer's Fixed Point Theorem in this metric space (*hint: A set that is convex and compact in* $\mathbb{R}$ *will be a interval set of the form* $[a, b]$ *where* $a, b \in \mathbb{R}$).

   Let $X = [a, b]$ where $a, b \in \mathbb{R}$. Since : $f[a, b] \to [a, b]$, then for $x \in [a, b]$, it follows that $f(x) \in [a, b]$. For x^* to be a fixed point, it follows that $x^* = f(x^*)$, or $f(x^*) - x^* = 0$. Consider a and b. Notice that $f(a) \geq a$ and $f(b) \leq b$, or $f(a) - a \geq 0$ and $f(b) - b \leq 0$. Since f is continuous, then we can apply the intermediate value theorem.

   First define a new function, $g(x) = f(x) - x$. Thus $g(a) = f(a) - a \geq 0$, and $g(b) = f(b) - b \leq 0$. By the intermediate value theorem, it follows that $g(b) \leq 0 \leq g(a)$. This leads to $f(x) - x = 0$, or $f(x) = x$. This is a fixed point.