PhD Mathcamp Final
Summer 2018
Time Limit: 120 Minutes

Name: ____________________

Instructions: Some questions on this test may be a bit difficult. Relax, and answer all 5 questions to the best of your ability (check every page to make sure you have answered everything). Note that partial solutions will receive partial credit, so putting something for a question will be better than leaving that question blank.

1. **(10 points)** Let $U = x^\alpha y^\beta$ where $(x, y) \in \mathbb{R}^2_+$, $\alpha + \beta = 1$, and $\alpha, \beta > 0$.[1]

   (a) Is U homogeneous? If so, of what degree?

   (b) Show that U is concave.

[1]Recall that $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$

2. **(5 points)** Express the following sets using set-builder notation $\{f(x) \in \mathbb{Z} : p(x)\}$, where $f(x)$ is a function of x, and $p(x)$ is a statement or condition of x.

   (a) {2,4,6,8,...}

   (b) {-27,-8,-1,0}

3. **(5 points)** Consider the metric space: $(\mathbb{R}, |\cdot|)$.

   (a) Show that the interval $(0, 1)$ is open.

   (b) Show that the set $\{1, 2, 3\}$ is closed.

4. **(10 points)** Prove the following:

   (a) The intersection of two convex sets is convex.

(b) The maximum of two convex functions is convex. In other words, if the functions f_1 and f_2 are convex, then $\max\{f_1, f_2\}$ is convex.

5. **(10 points)** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$. Now assume that there is a $\lambda \in (0, 1)$ such that:

$$|f(x) - f(x')| \leq \lambda |x - x'|$$

for all $x, x' \in \mathbb{R}$

Suppose we start with $y_1 \in \mathbb{R}$ and construct a sequence (y_n) by a applying the function f at each index to the previous element of the sequence. Thus our sequence would look like the following:

$$\begin{aligned}(y_n) &= (y_1,\ y_2,\quad y_3,\qquad y_4,\qquad ...) \\ &= (y_1, f(y_1), f(f(y_1)), f(f(f(y_1))), ...)\end{aligned}$$

Or in other words, $y_{n+1} = f(y_n)$.
You may find the following property of infinite series useful:

$$\sum_{i=1}^{\infty} ar^i = a \sum_{i=1}^{\infty} r^i = a \left(\frac{1}{1-r}\right)$$

where $a \in \mathbb{R}$ and $r \in (0, 1)$. In other words, this infinite sum is less than the constant: $a \left(\frac{1}{1-r}\right)$.

(a) Show that the sequence (y_n) is a Cauchy sequence.

(b) Since (y_n) is a Cauchy sequence, we see that (y_n) is a convergent sequence, or in other words there is a limit point y such that $\lim_{n\to\infty} y_n = y$. Prove that y is a fixed point of f.