Assignment 2 Solutions

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Set Theory

List out all the elements of each set, and put the elements within curly brackets { and }.

1. $A = \{n \in \mathbb{N} : n < 10\}$

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

2. $B = \{n \in \mathbb{Z} : n^2 < 6\}$

$$B = \{-2, -1, 0, 1, 2\}$$

3. $C = \{x \in \mathbb{R} : x^2 + 1 = 0\}$

$$C = \emptyset$$

Put the following sets in set builder notation. In other words, write each set in the form $\{x \in \mathbb{Z} : p(x)\}$, where $p(x)$ is an expression of x.

5. $D = \{-1, -2, -3, \ldots\}$

$$D = \{n \in \mathbb{Z} : n < 0\}$$

6. $E = \{-9, -4, -1, 0, 1, 4, 9\}$

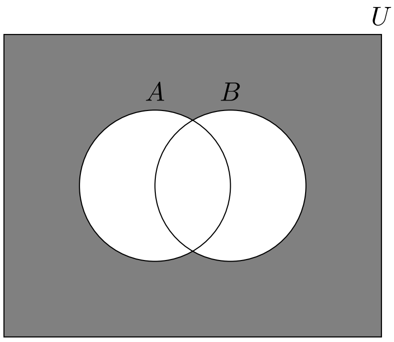
$$E = \left\{ \begin{cases} -(n^2) & \text{if } n \leq 0 \\ n^2 & \text{if } n > 0 \end{cases} : n \in \mathbb{Z} \text{ and } n^2 \leq 9 \right\}$$

7. $F = \{-1, 0, 1, 8, 27\}$

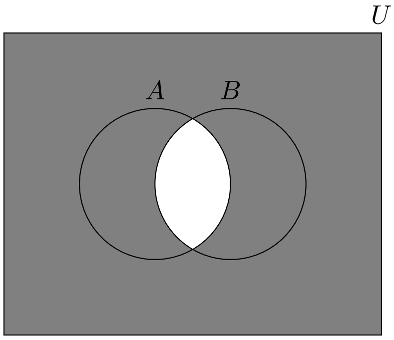
$$F = \{n^3 \in \mathbb{Z} : -1 \leq n \leq 3\}$$

Let U be a universal set, and let A and B be subsets of U. Draw a venn diagram for the following sets:

8. $\overline{A \cup B}$



9. $\overline{A} \cup \overline{B}$



10. Let $U = \{1, 2, 3, \ldots, 15\}$ be the universal set, $A = \{1, 3, 8, 9\}$, and $B = \{2, 8, 15\}$. Determine the following:

   (a) $A \cup B$

$$A \cup B = \{1, 2, 3, 8, 9, 15\}$$

   (b) $A \cap B$

$$A \cap B = \{8\}$$

   (c) $A - B$

$$A - B = \{1, 3, 9\}$$

   (d) $\overline{A}$

$$\overline{A} = \{2, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15\}$$

   (e) $A \cap \overline{B}$

$$A \cap \overline{B} = \{1, 3, 9\}$$

Direct Proof

11. Let $n \in \mathbb{Z}$. Prove that if n is even, then $3n - 11$ is odd.

   Proof. Assume n is even, thus $n = 2k$ where $k \in \mathbb{Z}$
   Therefore $3(2k) - 11 = 6k - 11 = 6k - 12 + 1 = 2(3k - 6) + 1$
   Since $(3k - 6) \in \mathbb{Z}$ then $3n - 11$ is odd.

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12. Let $x, y, z \in \mathbb{Z}$. Prove that if x and z are odd, then $xy + yz$ is even.

   Proof. Assume x and z are odd, thus $x = 2k + 1$ and $z = 2m + 1$ where $k, m \in \mathbb{Z}$
   Therefore $xy + yz = y(x + z) = y(2k + 1 + 2m + 1) = 2(my + ky + y)$
   Since $my + ky + y \in \mathbb{Z}$, $xy + yz$ is even.

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Proof by Contrapositive

13. Let $n \in \mathbb{Z}$. Prove that if $9x + 3$ is odd, then x is even.

   Proof. (Contrapositive) Suppose x is odd, then $x = 2k + 1$ for some $k \in \mathbb{Z}$
   Thus, $9x + 3 = 9(2k + 1) + 3 = 18k + 9 + 3 = 18k + 12 = 2(9k + 6)$
   Since $(9k + 6) \in \mathbb{Z}$, then $9x + 3$ is even.

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Proof by Contradiction

14. Prove that if x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$

Proof. (Contradiction) Let x and y are positive real numbers, and assume to the contrary that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

$$\begin{aligned}
\sqrt{x+y} &= \sqrt{x} + \sqrt{y} \\
\sqrt{x+y}\sqrt{x+y} &= \left(\sqrt{x} + \sqrt{y}\right)\sqrt{x+y} \\
x+y &= \sqrt{x(x+y)} + \sqrt{y(x+y)} \\
\sqrt{x^2} + \sqrt{y^2} &= \sqrt{x^2+xy} + \sqrt{y^2+xy}
\end{aligned}$$

Notice that since $xy > 0$, $\sqrt{x^2+xy} > \sqrt{x^2}$ and $\sqrt{y^2+xy} > \sqrt{y^2}$.
Thus, $\sqrt{x^2} + \sqrt{y^2} = \sqrt{x^2+xy} + \sqrt{y^2+xy}$ is a contradiction.

A simpler proof (like many of you did) is to assume $\sqrt{x+y} = \sqrt{x}+\sqrt{y}$, thus $x+y = x+y+2\sqrt{xy}$ $\Rightarrow \sqrt{xy} > 0$, which is a contradiction since $x, y > 0$.

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Proof by Cases

15. Let $a, b \in \mathbb{Z}$. Prove that $a - b$ is even if and only if a and b are of the same parity (either both are even or both are odd).

Since this is an iff statement, we have to prove both ways.

Proof. $\Rightarrow$) Assume that a and b are not of the same parity. We are required to prove that $a - b$ is odd
Without loss of generality, assume a is even and b is odd.
Thus $a = 2k$ and $b = 2j + 1$ from some $j, k \in \mathbb{Z}$
Therefore $a - b = 2k - 2j - 1 = 2k - 2j - 2 + 1 = 2(k - j - 1) + 1$.
Since $(k - j - 1) \in \mathbb{Z}$, it follows that $a - b$ is odd.

$\Leftarrow$) Assume that a and b are of the same parity. Now we are required to show that $a - b$ is even. We will proceed by using cases.

Case 1: Suppose a and b are even.
Thus $a = 2l$ and $b = 2m$ for some $l, m \in \mathbb{Z}$
So $a - b = 2l - 2m = 2(l - m)$
Since $(l - m) \in \mathbb{Z}$, then $a - b$ is even.

Case 2: Suppose a and b are odd.
Thus $a = 2p + 1$ and $b = 2q + 1$ for some $p, q \in \mathbb{Z}$
So $a - b = (2p + 1) - (2q + 1) = 2p - 2q = 2(p - q)$
Sicne $(p - q) \in \mathbb{Z}$, then $a - b$ is even.

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Mathematical Induction

16. Prove that

$$1 + 5 + 9 + 13 + \ldots + (4n - 3) = 2n^2 - n$$

for every $n \in \mathbb{N}$

Proof. (Proof by Induction) *Base case*: Consider $n = 1$. We need to show that $1 = 2(1)^2 - 1$:

$$\begin{aligned} 1 &= 2(1)^2 - 1 \\ 1 &= 2 - 1 \\ 1 &= 1 \end{aligned}$$

Thus the base case holds.

Inductive Step: We need to now show that $P(k) \Rightarrow P(k+1)$ holds.
We assume $P(k)$ is true, or in other words:

$$1 + 5 + 9 + 13 + \ldots + (4k - 3) = 2k^2 - k$$

We are required to prove that:

$$1 + 5 + 9 + 13 + \ldots + (4k - 3) + (4(k + 1) - 3) = 2(k + 1)^2 - (k + 1)$$

Observe that:

$$\begin{aligned} 1 + 5 + 9 + 13 + \ldots + (4k - 3) + (4(k + 1) - 3) &= 2k^2 - k + (4(k + 1) - 3) \\ &= 2k^2 + 3k + 1 \\ &= 2(k + 1)^2 - (k + 1) \end{aligned}$$

The result then follows by the Principle of Mathematical Induction.

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Relations and Functions

17. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + ax + b$ where $a, b \in \mathbb{R}$. Show that f is not one-to-one.

In order to show that this function is not one-to-one we need to find an example where the definition of one-to-one does not hold for this function. Consider $x = -a$ and $x = 0$. Notice that $f(-a) = f(0)$, thus the function is not one-to-one.

Set Theory Proofs

De Morgan's Laws are defined as such:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \tag{1}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \tag{2}$$

In order to show that two sets, X and Y, are equal, we need to show that every element of X is an Y, and every element of Y is in X. In other words, we need to show that $X \subseteq Y$ and $Y \subseteq X$. Suppose I wanted to show that the first expression was true, in other words that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. To show the equality holds, I need to show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. I'll first show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Proof Assume that $x \in \overline{A \cup B}$.
$\Rightarrow x \notin (A \cup B)$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in \overline{A}$ and $x \in \overline{B}$
$\Rightarrow x \in \overline{A} \cap \overline{B}$

Next, in order to show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, we would assume an arbitrary $x \in \overline{A} \cap \overline{B}$, and show that $x \in \overline{A \cup B}$. Although essential to the proof, I have left this part out.

18. Prove that expression (2) holds. Namely, that $\overline{A \cap B} = \overline{A} \cup \overline{B}$. Use the technique outlined above.

    Proof. To show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$, we need to show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$. We will first show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$:

    $\subseteq$) Suppose that $x \in \overline{A \cap B}$
    $\Rightarrow x \notin A \cap B$
    $\Rightarrow x \notin A$ or $x \notin B$
    $\Rightarrow x \in \overline{A}$ or $x \in \overline{B}$
    $\Rightarrow x \in \overline{A} \cup \overline{B}$
    Thus $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

    $\supseteq$) Suppose that $y \in \overline{A} \cup \overline{B}$
    $\Rightarrow y \in \overline{A}$ or $y \in \overline{B}$
    $\Rightarrow y \notin A$ or $y \notin B$
    $\Rightarrow y \notin A \cap B$
    $\Rightarrow y \in \overline{(A \cap B)}$
    Therefore $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

    Hence $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

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19. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ using the technique outlined above.

    Proof. $\subseteq$) Suppose that $x \in A \cap (B \cup C)$
    $\Rightarrow x \in A$ and $x \in (B \cup C)$
    $\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$
    $\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$
    $\Rightarrow x \in (A \cap B) \cup (A \cap C)$
    Thus $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

    $\supseteq$) Suppose that $y \in (A \cap B) \cup (A \cap C)$
    $\Rightarrow (y \in A$ and $y \in B)$ or $(y \in A$ and $y \in C)$
    $\Rightarrow y \in A$ and $(y \in B$ or $y \in C)$
    $\Rightarrow y \in A \cap (B \cup C)$
    Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

    Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

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