Assignment 3

Joe Patten

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1. Using the definition of convergence of a sequence prove that the following sequence converges to the proposed limit in $\mathbb{R}$:

$$\lim \frac{2}{\sqrt{n+3}} = 0$$

2. Consider the metric space $(\mathbb{R}, |\cdot|)$[1]. Prove that convergent sequence is a Cauchy sequence.

3. Consider the metric space $(\mathbb{R}, |\cdot|)$. Using the definition of open ball (or ε-neighborhood), prove that the interval $(0, 1)$ is open.

4. Consider the metric space $(\mathbb{R}, |\cdot|)$. Let:

$$B = \left\{ \frac{(-1)^n n}{n+1} : n \in \mathbb{N} \right\}$$

   (a) Find the limit points of B.
   (b) Is B a closed set?
   (c) Is B an open set?
   (d) Does B contain any isolated points?

5. Find the total differential for the following function:

$$z = 2x \sin y - 3x^2 y^2$$

6. Let $w = x^2 y - y^2$ where $x = \sin t$ and $y = e^t$.
   (a) Find $\frac{dw}{dt}$.
   (b) Evaluate $\frac{dw}{dt}$ at $t = 0$.

7. Consider the following coefficient matrix:

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

   (a) Determine the definiteness of the matrix above.
   (b) Convert the coefficient matrix into quadratic form.
   (c) Is the function convex or concave?

8. Consider the function $f(x) = \ln(1 + x)$.
   (a) Calculate $f(.5)$.

[1] The metric $|\cdot|$ is defined as the absolute value of the difference between two points. This metric is described in the notes as d_1

(b) Using a first order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

(c) Using a second order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

(d) Using a third order Taylor polynomial, approximate $f(.5)$ using $x_0 = 0$.

9. Differentiate implicitly to find $\frac{dy}{dx}$:

$$x^2 - 3xy + y^2 - 2x + y - 5 = 0$$

10. Use integration by parts to to evaluate the following integrals:

(a)

$$\int x \cos(x) dx$$

(b)

$$\int x e^{x^2} dx$$

11. Consider the Cobb-Douglas production function: $f(K, L) = AK^a L^b$ where $K, L \geq 0$, and $A > 0$.

(a) What conditions on a and b must be true in order for the function to be (weakly) concave (*Hint*: consider the Hessian matrix)?

(b) What conditions on a and b must be true in order for the function to be strictly concave?

12. In microeconomic theory, a budget set or opportunity set, is the set of all possible consumption bundles that an individual can afford given the prices of goods, $\mathbf{p}$, and that individual's income, y. The $n \times 1$ commodity vector, $\mathbf{x}$, is a list of amounts of different commodities. The price vector, $\mathbf{p}$, is an $n \times 1$ vector that tells the price for each commodity. The budget set B is defined as:

$$B = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p}^T \mathbf{x} \leq y\}$$

Show B is convex (using the definition of set convexity).[2]

[2] $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, ..., n\}$